THE GEOMETRY OF SYMMETRIC LOOPS

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Abstract

This paper is intended to deal with the geometry of locally analytic loops. In it we will show which loops generate symmetric spaces, the connection between the constructions of Sabinin and Loos, and examples of symmetric loops. The main results of this article were published in 1972 (Karanda 1972).

Introduction

For a long time, the role of quasigroups and loops in differential geometry was ambiguous. They were only applied in theories of projective planes and of nets. Recently (Loos 1969) it was shown that a symmetric space is a manifold Q with a differential mapping :—

$\mu: Q \times Q \to Q, \ \mu(x,y) = x.y = S_x y$ and properties :

1) $x.(y.z) = (x.y).(x.z)$
2) $x.(x.y) = y$
3) $x.x = x$
4) every point x has an neighbourhood Ux : $x.y = y$ implies that $y = x$ for all y in Ux.

This construction is reversible. S_x is Cartan's symmetry about point x and by the property 2), the property 1) becomes

$$S_{S_x y} = S_x S_y S_x \qquad (1)$$

but Loos never used the term "quasigroup" and paid less attention to the algebraic meaning of his binary operation. His construction is not suitable for an arbitrary homogeneous space.

Furthermore, it was shown that a left homogeneous space can be considered as a left loop and a left loop as a left homogeneous space (Sabinin 1972).

Symmetric Loops

Let Q(o) be a locally analytic loop (Mal'cev 1955) with mappings $S : x \to x^{-1}$ and $R : x \to xox$, where x^{-1} is the right inverse of x, $x \in Q$ and the operation (o) is introduced locally.

We define $L_x y = xoy$

Definition

A locally analytic loop Q(o) is called a symmetric loop if, for all x, y ε Q, the following conditions are fulfilled :

a) $S(xoy) = SxoSy$, that is, S is an automorphism.

b) $S^2 = id$, that is, S is an involutive.

c) $S\ell(x,y) = \ell(x,y)S$, that is, S commutes with $\ell(x,y) = L^{-1}_{(xoy)} L_x L_y$ which belongs to $as_\ell(Q)$, the associator of Q.

d) $xo(Sxoy) = y$ or $L_x^{-1} = L_{x-1} = L_{Sx}$, that is, the loop Q(o) has the left inverse property.

e) $xo(xoy) = (xox)oy$ or $L^2_x = L_x 2 = L_{Rx}$, that is, the loop Q(o) is left alternative.

f) The mapping R is locally a bijection.

Theorem 1

1. If Q (o) is a symmetric loop, then $L_Q/as_\ell(Q)$ is a symmetric space, where $L_Q = \{L_x\}_{x\varepsilon Q}$ is a group generated by all L_x, $x\varepsilon Q$.
2. If G/H is a symmetric space, then there exists a symmetric loop such that $G/H \simeq L_Q/as_\ell(Q)$

This theorem is proved as soon as we show the connection between the symmetric loop and the Loos quasigroup.

Remark

We define $S_x y = L_x S L_x^{-1} y$, then by the properties of the symmetric loop

$$x.y \overset{def}{=} S_x y = RxoSy \quad \text{or} \quad S_x = L_{Rx} S \tag{2}$$

$$\text{and } xoy = R^{-1}x.S_y \quad \text{or} \quad L_x = S_{R^{-1}x} S \tag{2'}$$

If y = e, then $x.e = S_x e = RxoSe = Rx = xox$ and if x = e, then $e.y = S_e y = Sy = y^{-1}$.

Theorem 2

The symmetric loop Q(o) with an isotope x.y = RxoSy is the Loos quasigroup Q(.).

Proof

The properties 2) and 3) are fulfilled : $x.x = S_x x = RxoSx =$ (xox) oSx = xo (xoSx) = xoe = x,

$$x.(x.y) = S_x S_x y = L_x S L_x^{-1} L_x S L_x^{-1} y = L_x S^2 L_x^{-1} y = y$$

We have to show that the quasigroup Q(.) connected with the symmetric loop Q(o) with the help of the isotope (2') is left distributive. By (2'), property 2), and RS = SR, the right and left parts of the equation (1) can be respectively rewritten as

$$S_x S_y S_x = S L_{Rx}^{-1} L_{Ry} L_{Rx}^{-1}, \quad S'_{S_x y} = S L_{R(SRxoy)}$$

Let S(Rx) = u, and as R is locally a bijection and $Rx \varepsilon Q(o)$ then

$$L_{R(uoy)} = L_u L_{Ry} L_u \tag{3}$$

By the property c) of the symmetric loop, we have

$$S L^{-1}_{(xoy)} L_x L_y S = L^{-1}_{(xoy)} L_x L_y, \quad \text{or} \quad L_{R(xoy)} = L_x L_{Ry} L_x$$

This equation coincides with (3).

It remains to show that locally x.y = y implies that y = x. In fact, from x.y = RxoSy, we have RxoSy = y or (xox) oSy = y, xo (xoSy) = y, $xoSy = x^{-1}oy$ and xoSy = S(xoSy). But it is easy to show that in a symmetric loop Sx = x only when x = e. Therefore xoSy = e and $Sy = x^{-1}$ That is y = x.

Theorem 3

The symmetric loop Q(o) is a special left Bol loop.

Proof

Using (3) we have

$$R\ (xoy)\ oz = xo\ (Ryo\ (xoz)\ ) \tag{4}$$

If z = e, then

$$R(xoy) = xo(Ryox) \tag{5}$$

Therefore by (5), (4) is rewritten as

$$(xo\ (Ryox)\ )\ oz = xo\ (Ryo\ (xoz)\ ).$$

Denoting Ry = u, we have the identity of a left Bol loop

$$(xo\ (uox))oz = xo(uo\ (xoz)) \tag{6}$$

It remains to show that Q(o) is special, that is for all $a,b,x,y \in Q$ $\ell(x,y)\,(aob) = \ell(x,y)\,ao\,\ell(x,y)\,b$. By the isotope (2') we have

$$\ell(x,y) = L^{-1}_{(xoy)} L_x L_y = S_{R^{-1}S(R^{-1}x.Sy)} S S_{R^{-1}x} S\, S_{R^{-1}y} S$$

But S_x and $S_e = S$ are automorphisms, and S commutes with $\ell(x,y)$.

Therefore

$$\ell(x,y)(aob) = \ell(x,y)\,(R^{-1}a.Sb) = \ell(x,y)R^{-1}a.S\ell(x,y)b.$$

Let us show that

$$R\ell(x,y)R^{-1} = \ell(x,y),$$

that is

$$R\ell(x,y)R^{-1}z = R(S(xoy)o(xo(yoR^{-1}z))).$$

By (5) we have

$$R\,\ell(xoy)R^{-1}z = S(xoy)o\{R(xo(yoR^{-1}z))oS(xoy)\} =$$

$$= S(xoy)o\{[xo((yo(zoy))ox]\,oS(xoy)\}.$$

By (6) we have

$$R\ell(x,y)R^{-1}z = S(xoy)o\,\{xo(yoz)\}$$

Thus $R\ell(x,y) = \ell(x,y)R$ and Q(o) is a special Bol loop.

Theorem 4

The Loos quasigroup Q(.) with isotope $xoy = R^{-1}x.Sy$ is the symmetric loop with a unit element e.

Proof

First we have to show that mapping $R{:}x \to x.e$ is invertible. The operation (o) has been introduced locally and hence the invertibility of R is connected with the possibility of solving the equation $S_x\, e = a$ about x, for all a. Locally points $e, a \in Q$ can be joined by a segment γ of a geodisic of a symmetric space uniquely (we have to confine ourselves within the normal neighbourhood of point e). Then a point p, which is the middle of the arc is the solution. That is p.e = a, according to the properties of Cartan's symmetry.

Let p' and p" be two different solutions. p'.e = p".e = a. Let us join point e with p' and p", with the help of two different geodesics γ' and γ'' respectively and then produce them up to point a. According to Cartan's symmetry, e,p', $S_{p'}e = a$ lie on one geodesic γ' and e,p", $S_{p''}e = a$ lie on the other geodesic γ''. And $\gamma' \neq \gamma''$ as $p' \neq p''$. This can not be because in the (normal) neighbourhood of point e, points e and a can be joined by a unique geodesic. Therefore $S_x e = a$ has a unique solution for all a (locally) and the mapping $R^{-1}{:}Q \to Q$ exists.

From the invertibility of R and S, it is obvious that Q(o) is a quasigroup. We notice that $Re = e$ and $R^{-1}e = e$. By the isotope (2') and the property 2) we have :

$eoy = e.(Sy) = e.(e.y) = y$ and $yoe = R^{-1}y.Se = R^{-1}y.e = R(R^{-1}y) = y.$

Therefore e is simultaneously a left and right unit in Q(o).
Furthermore,

$yoSy = R^{-1}y.y = R^{-1}y.R(R^{-1}y) = R^{-1}y.(R^{-1}y.e) = e.$

That is $Sy = y^{-1}$. We notice that $Syoy = SyoS(Sy) = e$ and$-^{1}y = Sy = y^{-1}$ for $y \varepsilon Q(o)$.

Let us show that S is an automorphism in Q(o) :

$SxoSy = R^{-1}Sx.y$, $S(xoy) = S(R^{-1}x.Sy) = e.(R^{-1}x.Sy) = (e.R^{-1}x).(e.Sy) = SR^{-1}x.y.$

But $R^{-1}S = SR^{-1}$ or $RS = SR$. Therefore $S(xoy) = SxoSy$ and S is an involutive automorphism in Q(o).

From

$x^{-1}o(xoy) = R^{-1}x^{-1}.(x^{-1}oSy) = R^{-1}x^{-1}.(R^{-1}x^{-1}y) = y,$

we see that the loop Q(o) has the left inverse property.

Let us show that $xox = Rx$:

$xox = R^{-1}x.Sx = R^{-1}x.(e.x) = (R^{-1}x.e).(R^{-1}x.x) = x.(xoSx) = x.e = Rx.$

This implies that $(xox)oy = Rxoy = x.Sy$. But $xo(xoy) = R^{-1}x.S(xoy) = R^{-1}x.(e.(xoy)) = (R^{-1}x.e).(R^{-1}x.(xoy)) = x.[R^{-1}x.(R^{-1}x.Sy)] = x.Sy.$

Therefore $(xox)oy = xo(xoy)$ and the loop Q(o) is left alternative.

It remains to show that $S\ell(x,y) = \ell(x,y)S$ for all $\ell(x,y)\,\varepsilon($ as$_{\ell}(Q)$.
In fact $S\ell(x,y) = SL^{-1}_{(xoy)}L_xL_y = L_{(xoy)}L^{-1}_xL^{-1}_y$.
The equation $S\ell(x,y) = \ell(x,y)S$ is equivalent to the equation

$$L_{(xoy)}L_x^{-1}L_y^{-1} = L^{-1}_{(xoy)}L_xL_y \quad \text{or} \quad L_{R(xoy)} = L_xL_{Ry}L_x$$

which concides with (3). If we denote $S_xy \overset{def}{=} x.y$ then equation (3) can be obtained from the property 1) of the Loos quasigroup. This completes the proof.

Theorem 1 now follows because the connection between the symmetric loop and the Loos quasigroup has been established.

Theorem 5

Let Q(o) be a symmetric loop. Then Q(o) is a symmetric space with a group of motions L_Q generated by $\{L_x : L_xy = xoy, x,y \varepsilon Q\}$ and with a stationary group $as_{\ell}(Q)$ generated by $\{\ell(x,y) = L^{-1}_{(xoy)} L_x L_y, x,y \varepsilon Q\}$.

The Cartan's symmetry about point e will be $S_e = S$.

Proof

It is easy to see that the group of motions of a symmetric space has a generating set $\{S_x S_e\}_{x \varepsilon Q}$. From the properties of the symmetric loop, we have $S_x = L_{Rx} S$, $L_x = S_R{}^{-1} .S$. And therefore the group generated by the operators $S_x S$ coincides with the group generated by the operators L_{Rx}

It was shown that $T \varepsilon L_Q$ means that $T = L_q \ell(x,y), q \varepsilon Q, \ell \varepsilon as_{\ell}(Q)$ (Sabinin 1971). This reference concerns the local analog of Sabinin's results. Therefore $Te = e$ implies that $L_q \, \ell(x,y)e = e = L_q \, e = e$ and $qoe = e$ implies that $q = e$, $L_q = id_q$, $T = \ell(x,y) \varepsilon \ \alpha s_{\ell}(Q)$: Thus $\alpha s_{\ell}(Q)$ is the stationary group of e in the symmetric space $Q(o)$ and $L_Q/\alpha s_{\ell}(Q(o))$ is a symmetric space. According to Loos' results, the symmetry about the point e is $S_e = S$. This completes the proof.

3. Symmetric Spaces of Rank 1 (Compact Type)

Let us construct models of the symmetric loops which determine symmetric spaces of rank 1: $SO(n+1)/SO(n)$, $SU(n+1)/S(U(n)xU(1))$, $Sp(n+1)/Sp(n)xSp(1)$, $F_4/SO(9)$. First we will use known models of symmetric spaces of rank 1 (for instance a sphere) and construct their respective Loos quasigroups and with the help of our results we will construct models of the symmetric loops.

(a) The Symmetric loop of the Space SO(n+1)/SO(n)

This symmetric space, the real elliptic space, is a sphere in an euclidean space E_{n+1}. But it was shown, that a sphere with the property

$$x.y = \frac{2(y,x)x}{(x,x)} - y \tag{7}$$

is a symmetric space (Loos 1969).

Let a unit $e \varepsilon Q(.)$. If $y = e$, then

$$S_x e = x.e = Rx = x.x = \frac{2(e,x)x}{(x,x)} - e = y_1 \tag{8}$$

From here we have

$$R^{-1}y_1 = x = \mu(y_1 + e) \tag{9}$$

μ can be found by inserting (9) in to (8) and then normalizing (9). That is,

$$\frac{2(e,y_1 + e)(y_1 + e)}{(y_1 + e,\ y_1 + e)} - e = y_1$$

and from linear independence of y_1 and e, we have :

$$2(e,y_1 + e) = (y_1 + e,\ y_1 + e) \tag{10}$$

which is fulfilled because $(e,e) = (y_1,\ y_1)$ on a sphere.

Normalizing (9) we have

$$\mu^2(y_1 + e,\ y_1 + e) = (y_1,\ y_1)$$

and by (10) we have

$$\mu = \pm\sqrt{\frac{(y_1,\ y_1)}{2(e,\ y_1 + e)}}$$

Basing our arguments on continuity we choose $\mu > 0$, and therefore

$$R^{-1}y_1 = \left[\frac{(y_1 y_1)}{2[(e,\ e) + (e,\ y_1)]}\right]^{\frac{1}{2}} (y_1 + e) \tag{11}$$

If x = e, then

$$S_e y = e.y = Sy = \frac{2(y,e)e}{(e,\ e)} - y \tag{12}$$

By (2'), (11) and (12), we have

$$xoy = R^{-1}x.Sy = \frac{2(Sy,R^{-1}x)}{(R^{-1}x,R^{-1}x)} R^{-1}x - Sy$$

After some calculations we have

$$xoy = \left[\frac{2(y,e)(e,x) - (y,e)(e,e) - (y,x)(e,e)}{(e,e)[(e,e) + (e,x)]}\right] x - \left[\frac{(y,x) + (y,e)}{(e,e) + (e,x)}\right] e + y$$

This formula can be simplified if for our sphere $(e,e) = 1$.

$$xoy = \left[\frac{2(y,e)(e,x) + (y,e) - (y,x)}{1 + (e,x)}\right]x - \left[\frac{(y,x) + (y,e)}{1 + (e,x)}\right]e + y$$

(b) The Symmetric loop of the space SU(n + 1)/S(U (n)xU(1))

For this symmetric space, the complex hermitian elliptic space, we can use the model of a sphere in the unitary space. But the group of motions of this sphere is wider than necessary. Therefore in order to come to the correct model, we have to regard the equivalence classes and not the points of our sphere, as points of our model, (points x and y are equivalent, $x \sim y$, if $y = e^{i\psi}\bar{x}$ where $\bar{x} = \overline{ox}$ – vector, O-center of a sphere). The set of these classes forms SU(n + 1)/S(U(n)xU(1)). If we denote the points of our model by X,Y,E, then (7) can be rewritten as

$$S_X Y = \frac{2(Y,X)}{(X,X)} X - Y \tag{13}$$

This formula correctly defines $S_{\bar{x}}$ if $\bar{x} \varepsilon X$, and if $S_X Y = Y$ then $X = Y$, (this is Cartan's symmetry on SU(n + 1)/S(U(x)xU(1)))

Let $\bar{x}\varepsilon X$, $\bar{y}\varepsilon Y$. (13) will be rewritten as :

$$\bar{x}.\bar{y} = \frac{2(\bar{y}.\bar{x})}{(\bar{x},\bar{x})}\bar{x} - \bar{y},$$

where $(\bar{x},\bar{y})$—hermitian scalar product. If $\bar{y} = \bar{e}$ then

$$\bar{x}.\bar{e} = \frac{2(\bar{e},\bar{x})}{(\bar{x},\bar{x})}\bar{x} - \bar{e} = R\bar{x} = B\bar{y} \tag{14}$$

where $|B| = 1$. And from this we have $R^{-1}\bar{y} = \mu(B\bar{y} + \bar{e})$

Using the same method as in (a), we have

$$B = \frac{(e,y)}{|(e,y)|} \quad \text{and} \quad \mu = \left[\frac{(y,y)}{2[(e,y)\,\beta^* + (e,e)]}\right]^{\frac{1}{2}} e^{i\psi}$$

Replacing $\bar{x}$, $\bar{y}$, $\bar{e}$ by X,Y,E respectively we have these formulae on our model.

If $\bar{x} = \bar{e}$, then $S_{\bar{e}}\,\bar{y} = \bar{e}.\bar{y} = S\bar{y} = 2\,\frac{(\bar{y},\bar{e})}{(\bar{e},\bar{e})}\bar{e} - \bar{y}$

Using our method, we have

$$XoY = \left[\frac{\{2|(E,X)|+(E,E)\}(Y,E)(E,X) - |(E,X)|(E,E)(Y,X0}{(E,E)\ [(E,X)|\{|(E,X)] + (E,E)\}}\right]X$$

$$- \left[\frac{(Y,X)(X,E) + |(E,X)|(Y,E)}{(E,X)^2 + |(E,X)|(E,E)}\right]E + Y$$

(c) The Symmetric loop of the space Sp(n + 1)/Sp(n)xSp(1)

We can realize this symmetric space, the quaternion hermitian elliptic space, on the unit sphere of the space K^{n+1} which has $(\overline{x},\overline{y}) = \sum_{i=1}^{n+1} x^i y*^i$ as scalar product (where K is the quaternion algebra). As in (b) the group of motions on the sphere, Sp(n) x Sp(1), is wider then necessary. Therefore we use the same method as in (b) and regard the equivalence classes as the points of our model. The set of these classes forms Sp(n + 1)/Sp(n) x Sp(1).

After analogous calculations to those in (a) and (b) we have:

$$q = \frac{(\overline{e},\overline{y})}{|(\overline{e},\overline{y})|} \quad , \quad \mu = \left[\frac{(\overline{y},\overline{y})}{2[|(\overline{e},\overline{y})| + (\overline{e},\overline{e})]}\right]^{\frac{1}{2}} q$$

where q — quaternion factor and $|q| = 1$. On our model, we have

$$XoY = \left[\frac{\{2|(E,X)| + (L,E)\}(Y,E)(E,X) - |(E,X)|(E,E)(Y,X)}{(E,E)\ |\ (E,X)\ |\{|(E,X)| + (E,E)\}}\right]X$$

$$- \left[\frac{(Y,X)(X,E) + |(E,X)|(Y,E)}{|(E,X)|^2 + |(E,X)|(E,E)}\right]E + Y$$

(d) The symmetric loop of the space $F_4/SO(9)$

Let us construct this space, the octave plane in the model of Freudenthal (Freudenthal 1951), M^+_3. The points x,y,e, of (7) are replaced by matrices $\hat{x},\hat{y},\hat{e}$ respectively :

$$S_{\hat{x}}\hat{y} = \frac{2(\hat{y},\hat{x})}{(\hat{y},\hat{y})}\hat{x} - \hat{y}$$

It is easy to show, that $S_{\hat{x}}$ is a Cartan's symmetry.

Using the same method as in (a) we have :

$$\mu = \left[\frac{(\hat{y},\hat{y})}{2(\hat{e},\hat{y}) + (\hat{e},\hat{e})}\right]^{\frac{1}{2}} \text{ and}$$

$$\hat{x}o\hat{y} = \left[\frac{2(\hat{y},\hat{e})(\hat{e},\hat{x}) + (\hat{y},\hat{e})(\hat{e},\hat{e}) - (\hat{y},\hat{e})(\hat{e},\hat{e})}{(\hat{e},\hat{e})\left[(\hat{e},\hat{e}) + (\hat{e},\hat{x})\right]}\right]\hat{x}$$

$$- \left[\frac{(\hat{y},\hat{x}) + (\hat{y},\hat{e})}{(\hat{e},\hat{x}) + (\hat{e},\hat{e})}\right]\hat{e} + \hat{y}$$

4. The Geodesic Loops of the Symmetric space

The Geodesic loops of the symmetric space of an affine connection were first constructed by Sabinin.

Let A_n be a smooth local space of an affine connection (we confine ourselves within a normal neighbourhood of the point e). The geodesic $\gamma(t)$ with a parameter t, joins the point e with the point y uniquely (in a normal neighbourhood) and $\gamma(0) = e, \gamma(1) = y$. With this, the choice of canonical parameter t is fixed, and $\dot{\gamma}(0) = \bar{\zeta}$ is the tangent vector to $\gamma(t)$ at the point e. The victor $\bar{\eta}$ is obtained by a parallel translation of $\bar{\zeta}$ to the point x along the geodesic arc $\alpha(t)$ which joins the point e with the point x. Moreover, $\gamma_1(t)$ is the geodesic with the canonical parameter t and $\gamma_1(0) = x, \dot{\gamma}_1(0) = \bar{\eta}, \gamma_1(1) = z$. Thus the geodesic $\gamma_1(t)$ with the parameter t has been determined uniquely. Therefore, we have $A_n \times A_n \to A_n$ (locally) or a binary operation z = xoy.

From the property of continuity, we have xoy $\to$ x when y = ι and xoe $\overset{def}{=}$ x. Therefore we have a smooth binary operation. It is easy to see that by z and x,y can be restored, that is, there exists a left division x/z = y. Besides this e is a two-sided unit, eoy = yoe = y. Thus $A_n(o)$ is a loop.

Theorem 6

The geodesic loop of a symmetric space is a symmetric loop,

Proof

Let $d(p,e) = d(p,x)$ be a distance. According to invariance of parallel translation by the action of symmetry, we have $S_p \gamma(-t) = \gamma_1(t)$. This is a well known fact from the theory of E. Cartan. Therefore in our case $S_p{}^x(-\bar{\xi}) = \bar{\eta}$ (where S^x is the transformation induced by S_p on the tangent space $S_p^x : T_e \to T_x$) or $S_p{}^x \bar{\zeta} = -\bar{\eta}$. The geodesic $\gamma_1(-t)$ which has tangent vector $-\bar{\eta}$ is unique. But $\gamma_1(t) = S_p(-t)$ and $\gamma(-t) = S_e \gamma(t)$. Therefore $S_p S_e \gamma(t) = \gamma_1(t)$ and $S_p S_e \gamma(1) = \gamma_1(1)$ i.e. $S_p S_e y = xoy$. We know that $S_p S_e = L_x$, where L_x is left translation of the symmetric space $(L_x y = x*y)$. Therefore $L_x y = xoy = x*y$ and the symmetric loop coincides with the geodesic loop.

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